Derivation of the Complementary Theorem from the Riemann-Roch Theorem.

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In his book* and subsequent memoirs† on the "Theory of the Algebraic Functions of a Complex Variable," Dr. J. C. Fields gives several proofs of the complementary theorem, of which theorem the well-known Riemann-Roch theorem is a particular case. The purpose of the present paper is not to give an independent proof of the complementary theorem, but rather to derive the same from the Riemann-Roch theorem.

As a preliminary we shall state the complementary theorem in its most general form. We shall suppose that the fundamental equation is

$$f(z, u) = u^n + u^{n-1} f_1(z) + \dots + f_n(z) = 0, \tag{1}$$

in which each $f_s(z)$ is a rational function of z. The only limitation on $f(z, u)$ is that it may not possess a repeated factor. The n expansions of u in the vicinity of a value $z = a_k$ group themselves into r_k cycles of orders

$$\nu_1^{(k)}, \quad \nu_2^{(k)}, \dots, \quad \nu_{r_k}^{(k)},$$

respectively. The expansions of a cycle of order $\nu_s^{(k)}$ proceed according to ascending integral powers of

$$(z - a_k)^{1/\nu_s^{(k)}}.$$

The n expansions of u in the vicinity of the value $z = \infty$ group themselves into r_∞ cycles of orders

$$\nu_1^{(\infty)}, \quad \nu_2^{(\infty)}, \dots, \quad \nu_{r_\infty}^{(\infty)},$$

respectively. The expansions of a cycle of order $\nu_s^{(\infty)}$ proceed according to ascending integral powers of

$$\left(\frac{1}{z}\right)^{1/\nu_s^{(\infty)}}.$$

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* Mayer and Muller, "Theory of the Algebraic Functions of a Complex Variable," Berlin (1916).

† One of the most recent and complete bears the title "On the Foundations of the Theory of Algebraic Functions of one Variable." *Phil. Trans. of the Royal Soc. of London*, Series A, Vol. CCXII, pp. 339–373.

Let us imagine we have a set of numbers, one to a cycle, in connection with all values of z. The nature of these numbers we require to be such that the number relating to a cycle of order $\nu_s^{(k)}$ in connection with a value $z=a_k$ is a multiple of $1/\nu_s^{(k)}$, while the number relating to a cycle of order $\nu_s^{(\infty)}$ in connection with the value $z=\infty$ is a multiple of $1/\nu_s^{(\infty)}$; moreover, only a finite number of such numbers may be different from zero. These numbers we agree to denote by $\tau_s^{(k)}$, $\tau_s^{(\infty)}$, respectively, and the set by (τ). Such a set (τ) is called a basis and the numbers $\tau_s^{(k)}$, $\tau_s^{(\infty)}$ are called elements of the basis. Those elements of (τ) relative to finite values of z and to the value $z=\infty$ constitute the partial bases $(\tau)'$ and $(\tau)^\infty$, respectively. The order of coincidence of $G(z, u)$, a rational function of (z, u), relative to a cycle of order $\nu_s^{(k)}$ in connection with a value $z=a_k$ is defined to be the least number appearing as a power of the element of expansion $z-a_k$ in the expression obtained on replacing u in $G(z, u)$ by any of the expansions of u relative to such cycle. The order of coincidence of $G(z, u)$ relative to a cycle in connection with the value $z=\infty$ is defined in a corresponding manner, where, of course, the element of expansion is $1/z$. Suppose now that $G(z, u)$ is a specific rational function of (z, u) not identically zero for any of the branches. Then its orders of coincidence form a basis (γ). The partial bases

$$\left(-\tau+\gamma-1+\frac{1}{\nu}\right)', \quad \left(-\tau+\gamma+1+\frac{1}{\nu}\right)^\infty$$

constitute a basis $(\bar{\tau})$ complementary adjoint to the basis (τ) to the level furnished by the function $G(z, u)$. A rational function of (z, u) is built on a basis (τ) if each of its orders of coincidence is equal to or greater than the corresponding element of the basis (τ). Denoting the general rational function of (z, u) built on a basis (τ) by $R_\tau(z, u)$ and the number of arbitrary constants appearing therein by N_τ, the complementary theorem consists in the formula

$$N_\tau+\tfrac{1}{2}\sum_k \sum_{s=1}^{r_k} \tau_s^{(k)}\nu_s^{(k)}=N_{\bar{\tau}}+\tfrac{1}{2}\sum_k \sum_{s=1}^{r_k} \bar{\tau}_s^{(k)}\nu_s^{(k)}. \tag{2}$$

Moreover, for the purpose of the present paper this formula will be regarded as affording a statement of the Riemann-Roch theorem, where for such purpose, however, equation (1) must be irreducible, the specific rational function of (z, u) used as a level in determining a basis $(\bar{\tau})$ complementary adjoint to (τ) must be $f'_u(z, u)$ and all the elements in one of the bases (τ), $(\bar{\tau})$ must be zero or negative. This statement of the Riemann-Roch theorem does not differ essentially from the usual one.

We now propose to derive the complementary theorem from the Riemann-Roch theorem. For the present we shall suppose that equation (1) is irredu-

cible and that (τ), $(\bar{\tau})$ are complementary adjoint to the level furnished by the function $f'_u(z,u)$. It may happen that not both N_τ, $N_{\bar{\tau}}$ are zero; in that case we suppose that N_τ is different from zero. Then $R_\tau(z,u)$ does not vanish identically, and, consequently, particular values can be ascribed to the N_τ arbitrary constants appearing therein, in such a way as to insure that the resulting specific function $R(z,u)$ is not zero identically. But the orders of coincidence of $R(z,u)$ form a basis (ρ). The bases $(\tau-\rho)$, $(\bar{\tau}+\rho)$ are complementary adjoint to the level furnished by the function $f'_u(z,u)$ and all the elements in the former of these are zero or negative. The Riemann-Roch theorem is, therefore, applicable and hence we may write

$$N_{\tau-\rho}+\tfrac{1}{2}\sum_k\sum_{s=1}^{r_k}(\tau_s^{(k)}-\rho_s^{(k)})\nu_s^{(k)}=N_{\bar{\tau}+\rho}+\tfrac{1}{2}\sum_k\sum_{s=1}^{r_k}(\bar{\tau}_s^{(k)}+\rho_s^{(k)})\nu_s^{(k)}. \tag{3}$$

The general rational functions of (z,u) built on the bases $(\tau-\rho)$, $(\bar{\tau}+\rho)$ are evidently $R_\tau(z,u)/R(z,u)$, $R_{\bar{\tau}}(z,u)R(z,u)$, respectively, and consequently,

$$N_{\tau-\rho}=N_\tau,\quad N_{\bar{\tau}+\rho}=N_{\bar{\tau}}. \tag{4}$$

Moreover, the elements of the basis (ρ) are connected by the relation

$$\sum_k\sum_{s=1}^{r_k}\rho_s^{(k)}\nu_s^{(k)}=0. \tag{5}$$

On utilizing formulae (4), (5) to effect a simplification of formula (3), the result is the complementary formula (2).

Still supposing that equation (1) is irreducible and that (τ), $(\bar{\tau})$ are complementary adjoint to the level furnished by the function $f'_u(z,u)$ it remains to treat the case in which N_τ, $N_{\bar{\tau}}$ are both zero. Suppose that (t) is a basis each element of which is equal to or less than the corresponding element of the basis (τ) and suppose, moreover, that N_t is different from zero. Denote by $(\bar{t})$ the basis complementary adjoint to (t) to the level furnished by the function $f'_u(z,u)$. We may, therefore, utilize the result of the previous paragraph in its application to the bases (t), $(\bar{t})$ to arrive at the formula

$$N_t+\tfrac{1}{2}\sum_k\sum_{s=1}^{r_k}t_s^{(k)}\nu_s^{(k)}=N_{\bar{t}}+\tfrac{1}{2}\sum_k\sum_{s=1}^{r_k}\bar{t}_s^{(k)}\nu_s^{(k)}. \tag{6}$$

But since in all cases

$$t_s^{(k)}+\bar{t}_s^{(k)}=\tau_s^{(k)}+\bar{\tau}_s^{(k)}, \tag{7}$$

and because each element $t_s^{(k)}$ is equal to or less than the corresponding element $\tau_s^{(k)}$, it follows that each element $\bar{t}_s^{(k)}$ is equal to or greater than the corresponding element $\bar{\tau}_s^{(k)}$ and, consequently,

$$N_{\bar{t}}=0. \tag{8}$$

By utilizing formulae (7), (8) we can give to formula (6) the form

$$\tfrac{1}{2}\sum_k \sum_{s=1}^{r_k} (\tau_s^{(k)} - \bar{\tau}_s^{(k)}) \nu_s^{(k)} = \sum_k \sum_{s=1}^{r_k} (\tau_s^{(k)} - t_s^{(k)}) \nu_s^{(k)} - N_t. \tag{9}$$

Now, the number of conditions to be applied to $R_t(z, u)$ to reduce it to $R_\tau(z, u)$ is certainly not more than

$$\sum_k \sum_{s=1}^{r_k} (\tau_s^{(k)} - t_s^{(k)}) \nu_s^{(k)}$$

Since, however, the number of these conditions is N_t, it follows that the right side of (9) is not negative. That is,

$$\tfrac{1}{2}\sum_k \sum_{s=1}^{r_k} \tau_s^{(k)} \nu_s^{(k)} - \tfrac{1}{2}\sum_k \sum_{s=1}^{r_k} \bar{\tau}_s^{(k)} \nu_s^{(k)} \tag{10}$$

is not negative. By interchanging the rôles of the bases (τ), $(\bar{\tau})$, it follows in a similar manner that the expression in (10) is not positive. Therefore,

$$\tfrac{1}{2}\sum_k \sum_{s=1}^{r_k} \tau_s^{(k)} \nu_s^{(k)} = \tfrac{1}{2}\sum_k \sum_{s=1}^{r_k} \bar{\tau}_s^{(k)} \nu_s^{(k)}. \tag{11}$$

This is, however, the complementary formula for the case under discussion, since N_τ, $N_{\bar{\tau}}$ are both zero. This completes the derivation of the complementary theorem from the Riemann-Roch theorem for the case in which equation (1) is irreducible, and in which $(\bar{\tau})$ is complementary adjoint to (τ) to the level furnished by the function $f'_u(z, u)$.

We shall denote by $(\tau) - \dfrac{1}{\nu}$ a basis, the elements of which with a single exception are all equal to the corresponding elements of a basis (τ) and for the excepted cycle of order ν the element is $\dfrac{1}{\nu}$ less than the corresponding element of (τ). Substituting in the above sentence for the word less the word greater we have the definition of a basis denoted by $(\tau) + \dfrac{1}{\nu}$. If (τ), $(\bar{\tau})$ are complementary adjoint to the level furnished by a specific rational function of (z, u) not identically zero for any of the branches, it is evident that $(\tau) - \dfrac{1}{\nu}$, $(\bar{\tau}) + \dfrac{1}{\nu}$ are complementary adjoint to the same level. If equation (1) is irreducible, and if the specific rational function of (z, u) used as level in determining a basis $(\bar{\tau})$ complementary adjoint to a given basis (τ) is $f'_u(z, u)$, then from the complementary formulae as already established relative to the

complementary bases (τ), $(\bar{\tau})$ and $(\tau)-\dfrac{1}{\nu}$, $(\bar{\tau})+\dfrac{1}{\nu}$, it readily follows that

$$(N_{\tau-\frac{1}{\nu}}-N_{\tau})+(N_{\bar{\tau}}-N_{\bar{\tau}+\frac{1}{\nu}})=1. \tag{12}$$

This formula is equivalent to the statement that of the two functions $R_{\tau-\frac{1}{\nu}}(z,u)$, $R_{\bar{\tau}}(z,u)$ one and only one possesses relative to the single cycle of order ν the precise order of the basis on which it is built.

Still supposing that equation (1) is irreducible, if any specific rational function of (z,u) not identically zero is used as level in determining a basis $(\bar{\tau})$ complementary adjoint to a given basis (τ) it is plain that formula (12) holds good, since the quotient of $R_{\bar{\tau}}(z,u)$ by the specific rational function used as level in determining $(\bar{\tau})$, is the same whatever be the function employed as level, or in other words $N_{\bar{\tau}}$ is the same for all such bases $(\bar{\tau})$.

Finally, we shall prove that formula (12) is true, should equation (1) happen to be reducible. In that case suppose that (τ), $(\bar{\tau})$ are complementary adjoint to the level furnished by a specific rational function $G(z,u)$ not identically zero for any of the branches. Now,

$$(N_{\tau-\frac{1}{\nu}}-N_{\tau})+(N_{\bar{\tau}}-N_{\bar{\tau}+\frac{1}{\nu}}) \tag{13}$$

must possess one of the values 0, 1 or 2. Suppose that the single cycle of order ν for which $(\tau)-\dfrac{1}{\nu}$ differs from (τ) relates to the equation $f_i(z,u)=0$, which is any one of the irreducible equations comprised under equation (1). Denote by $R_{\tau}^{(i)}(z,u)$ the general rational function of (z,u) conditioned by that part of a basis (τ) relating to the irreducible equation $f_i(z,u)=0$. We may then employ formula (12) as already arrived at, to conclude that one of the functions $R^{(i)}_{\tau-\frac{1}{\nu}}(z,u)$, $R^{(i)}_{\bar{\tau}}(z,u)$ possesses for the single cycle of order ν the precise order of the basis on which it is built. Writing $f(z,u)$ in the form $f_i(z,u)Q_i(z,u)$ and denoting the reduced form, relative to $f_i(z,u)=0$ of $R^{(i)}_{\tau}(z,u)/Q_i(z,u)$ by $H^{(i)}_{\tau}(z,u)$, it is plain that $H^{(i)}_{\tau-\frac{1}{\nu}}(z,u)Q_i(z,u)$, $H^{(i)}_{\bar{\tau}}(z,u)Q_i(z,u)$ are built on the bases $(\tau)-\dfrac{1}{\nu}$, $(\bar{\tau})$, respectively, relative to $f(z,u)=0$, and one of these functions possesses for the single cycle of order ν the precise order of the basis on which it is built. The same is, therefore, true for at least one of $R_{\tau-\frac{1}{\nu}}(z,u)$, $R_{\bar{\tau}}(z,u)$. In other words, expression (13) must equal 1 or 2. It is, however, not possible that it should possess the value 2 for then $R_{\tau-\frac{1}{\nu}}(z,u)R_{\bar{\tau}}(z,u)/G(z,u)$ would be a function with the sum of its

residues equal to ν times an arbitrary constant. Consequently, expression (13) equals 1 in all cases.

Just in passing, the writer wishes to remark that this result may be employed even more conveniently than the complementary theorem, of which it is a corollary, to demonstrate the existence of Abelian integrals of the third kind; also if equation (1) is irreducible, formula (12) may be employed to prove that the general ϕ-function is either zero identically or possesses for any cycle the order of coincidence required by the ϕ-basis; finally, if equation (1) is irreducible, it follows readily from formula (12) that a rational function of (z, u) can be constructed with exactly one pole, when and only when the general ϕ-function is identically zero.

We now proceed to derive the general complementary theorem from the generalized theorem contained in formula (12). Before doing so we shall state formula (12) in the form

$$(N_{\tau-\frac{1}{\nu}}-N_{\bar{\tau}+\frac{1}{\nu}})=(N_{\tau}-N_{\bar{\tau}})+1. \tag{14}$$

The bases (τ), $(\bar{\tau})$ may be written in the forms

$$(\bar{\tau})=(\tau)-(h)+(g), \quad (\tau)=(\bar{\tau})+(h)-(g), \tag{15}$$

in which (h), (g) are bases, all the elements of which are zero or positive. By repeated application of formula (13) to the proper complementary bases and by addition of the results, we get the formulae

$$\left.\begin{aligned}(N_{\tau-h}-N_{\bar{\tau}+h})&=(N_{\tau}-N_{\bar{\tau}})+\sum_k\sum_{s=1}^{r_k}h_s^{(k)}\nu_s^{(k)},\\ (N_{\bar{\tau}-g}-N_{\tau+g})&=(N_{\bar{\tau}}-N_{\tau})+\sum_k\sum_{s=1}^{r_k}g_s^{(k)}\nu_s^{(k)}.\end{aligned}\right\} \tag{16}$$

The left sides of these two equations are, however, equal. Hence,

$$2(N_{\tau}-N_{\bar{\tau}})=\sum_k\sum_{s=1}^{r_k}g_s^{(k)}\nu_s^{(k)}-\sum_k\sum_{s=1}^{r_k}h_s^{(k)}\nu_s^{(k)}. \tag{17}$$

Replacing the right side of equation (17) by its equivalent value

$$\sum_k\sum_{s=1}^{r_k}\bar{\tau}_s^{(k)}\nu_s^{(k)}-\sum_k\sum_{s=1}^{r_k}\tau_s^{(k)}\nu_s^{(k)},$$

and dividing by 2 throughout, the result is the complementary formula, arrived at under the most general conditions.

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